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On travelling waves and double-periodic structures in two-dimensional sine–Gordon systems

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Abstract. Exact travelling-wave solutions of the $(2+1)$-dimensional sine–Gordon equation possessing a velocity smaller than the velocity of the linear waves in the correspondent model system are obtained. The dependence of their dispersion relations and allowed areas for the wave parameters on the wave amplitude are discussed. The obtained waves contain as particular cases static structures consisting of elementary cells with zero topological charge. The self-consistent parameters of one static structure are calculated. The obtained structures require minima spatial system sizes for their existence. As an illustration the obtained results are applied for a description of structures in spin systems with an anisotropy created by a magnetic field or by a crystal anisotropy field.

1. Introduction

The sine–Gordon equation is one of the most famous nonlinear partial differential equations because of its soliton solutions [1] and its wide application in describing different physical systems, for example the local electrodynamics of the Josephson junctions or bounded vortex states below the critical temperature for the Kosterlitz–Thouless phase transitions in spin systems with an anisotropy, created by an external magnetic field or by a crystal anisotropy field [2–5]. There are two kinds of sine–Gordon model systems: pure dispersive systems and systems with dissipative losses. Various methods exist for obtaining exact analytical expressions for travelling and standing waves in pure dispersive $(1+1)$-dimensional systems: the inverse scattering transform [6], the Hirota method [7], Lamb's ansatz [8–12] etc [13, 14]. The problem of obtaining exact analytical solutions for systems including dissipative losses is more complex. Such a solution exists in the $(1+1)$-dimensional case [15], but the usual way to take into account the above effects is the application of perturbation theory if the effects are small [16, 17] or numerical investigation of the solutions of the model equation if the effects are large [18, 19].

In this article we discuss the $(2+1)$-dimensional sine–Gordon equation without terms corresponding to dissipative losses

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\phi}{\partial t^2} = \sin\phi(y, z, t). \tag{1}$$

There exists an approach for obtaining exact analytical solutions of equation (1) [20]. The approach is based on the Lamb's ansatz [8] and the resulting solutions are travelling waves of the kind

$$\phi = 4\tan^{-1}[Af(\alpha y; k_1)g(\beta z + \delta\gamma t; k_2)] \tag{2}$$

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where f and g are elliptic Jacobi functions [21] whose generation equations are

$$\left(\frac{\mathrm{d}f}{\mathrm{d}y}\right)^2 = a_1 f^4 + b_1 f^2 + c_1 \tag{3}$$

$$\left(\frac{\mathrm{d}g}{\mathrm{d}\xi}\right)^2 = a_2 g^4 + b_2 g^2 + c_2 \tag{4}$$

where $\xi = \beta z + \delta\gamma t$ and a_i, b_i, c_i, $i = 1, 2$, are parameters depending on the modulus k_1 of the corresponding Jacobi elliptic function. $\delta = \pm 1$ and the solution parameters α, β, γ, A, k_1 and k_2 are connected through the following three algebraic relations

$$\alpha^2 b_1 - (\gamma^2 - \beta^2) b_2 = 1 \tag{5a}$$

$$\alpha^2 a_1 + (\gamma^2 - \beta^2) A^2 c_2 = 0 \tag{5b}$$

$$\alpha^2 A^2 c_1 + (\gamma^2 - \beta^2) a_2 = 0 \tag{5c}$$

and restricted by the inequalities

$$0 \leqslant k_i \leqslant 1 \qquad i = 1, 2. \tag{6}$$

The properties of the nonlinear waves (2) strongly depend on the ratio between their phase velocity v and the phase velocity v_l of the linear waves in the investigated system (if for an example the low T_c Josephson junction is considered v_l is known as the velocity of Swihart [22, 23]). Tachyonic ($v/v_l > 1$) quasi-one-dimensional kink solutions of the $(2+1)$-dimensional sine–Gordon equation are well known [24]. The tachyonic waves of kind (2) contain as particular cases other well known one-dimensional waves—the plasma, breather and fluxon waves [15]. If $v/v_l = 1$ the possible waves of kind (2) have some similarities with the surface waves [25]. The last class of waves (2) contains solutions with velocity $v < v_l$ and some of the spatial properties of these waves are connected to the function f. If the function f is a constant then the waves are reduced to the one-dimensional kink solution of the sine–Gordon equation. If the function f is not a constant the waves require two spatial dimensions for their existance. If $v = 0$ the waves (2) are reduced to two kinds of static distributions which contain as particular cases the one-dimensional distribution of Ferrel and Prange [26] and consist of rectangular elementary cells. The waves leading to static distributions with periodical behaviour of ϕ on the boundaries of the correspondent static distributions are discussed in [27]. The possible waves (2) leading to static distributions with $\phi = 0$ on the elementary cell boundaries will be discussed in this paper.

The basis of the investigation are the twelve main elliptic Jacobi functions [21]. The investigated waves are described in section 2. In section 3 the dispersion relations and the allowed and forbidden areas for the wavenumbers and frequency are obtained. The influence of the boundary conditions on the parameters of one static structure is investigated. As an illustration the connection between the static structures and the bounded vortex states in spin systems with an anisotropy are also discussed.

2. The waves and their parameters

Using table 1 which gives the connections between the coefficients a, b and c in the generating equations for the elliptic Jacobi functions and the elliptic modulus k, the following waves of kind (2) with velocity $v < v_l$ can be obtained

$$\phi_1 = 4\tan^{-1}[A\,\mathrm{cn}(\alpha y; k_1)/\mathrm{cn}(\beta z + \delta\gamma t; k_2)] \tag{7a}$$

Table 1. Relations between the modules of the twelve main elliptic Jacobi functions and the coefficients a, b and c in their generation equations.

Elliptic function	a	b	c
cn	$-k^2$	$2k^2-1$	$1-k^2$
sn	k^2	$-1-k^2$	1
dn	-1	$2-k^2$	k^2-1
sn/cn	$1-k^2$	$2-k^2$	1
cn/sn	1	$2-k^2$	$1-k^2$
sn/dn	$-k^2(1-k^2)$	$2k^2-1$	1
dn/sn	1	$2k^2-1$	$-k^2(1-k^2)$
cn/dn	k^2	$-1-k^2$	1
dn/cn	1	$-1-k^2$	k^2
1/sn	1	$-1-k^2$	k^2
1/cn	$1-k^2$	$2k^2-1$	$-k^2$
1/dn	k^2-1	$2-k^2$	-1

with the parameters

$$k_1^2 = A^2[\alpha^2(1+A^2)+1]/[\alpha^2(1+A^2)^2] \qquad (7b)$$

$$k_2^2 = [(\beta^2-\gamma^2)(1+A^2)+A^2]/[(\beta^2-\gamma^2)(1+A^2)^2] \qquad (7c)$$

$$(\beta^2-\gamma^2)-\alpha^2 = (1-A^2)/(1+A^2). \qquad (7d)$$

The following particular case is possible for this wave

$$k_1 = k_2 = 1.$$

Then

$$\phi_{1,1} = 4\tan^{-1}[A\cosh(\beta z+\delta\gamma t)/\cosh(\alpha y)] \qquad (8a)$$

with parameters

$$\alpha = [A^2/(A^2+1)]^{1/2} \qquad (8b)$$

$$\gamma = [\beta^2-1/(A^2+1)]^{1/2}. \qquad (8c)$$

The next possible combination of elliptic Jacobi functions is

$$\phi_2 = 4\tan^{-1}[A\,\mathrm{sn}(\alpha y;k_1)\mathrm{sn}(\beta z+\delta\gamma t;k_2)/\mathrm{cn}(\beta z+\delta\gamma t;k_2)] \qquad (9a)$$

with parameters

$$k_1^2 = A^2[\alpha^2(1-A^2)+1]/[\alpha^2(1-A^2)] \qquad (9b)$$

$$k_2^2 = 1-A^2[(\beta^2-\gamma^2)(1-A^2)-1]/[(\beta^2-\gamma^2)(1-A^2)] \qquad (9c)$$

$$(\beta^2-\gamma^2)-\alpha^2 = 1/(1+A^2). \qquad (9d)$$

The possible particular case here is

$$k_1 = 1 \qquad k_2 = (1-A^4)^{1/2}$$

and the corresponding wave is

$$\phi_{2,1} = 4\tan^{-1}[A\tanh(\alpha y)\mathrm{sn}(\beta z+\delta\gamma t;k_2)/\mathrm{cn}(\beta z+\delta\gamma t;k_2)] \qquad (10a)$$

where

$$\alpha = A/(1-A^2) \qquad (10b)$$

$$\gamma = [\beta^2-1/(1-A^2)^2]^{1/2}. \qquad (10c)$$

The third wave is

$$\phi_3 = 4\tan^{-1}[A\,\mathrm{sn}(\alpha y; k_1)\mathrm{cn}(\beta z + \delta\gamma t; k_2)/\mathrm{sn}(\beta z + \delta\gamma t; k_2)]. \tag{11a}$$

The parameters of the wave are

$$k_1^2 = A^2[\alpha^2(A^2-1)-1]/[\alpha^2(A^2-1)] \tag{11b}$$
$$k_2^2 = [A^2 + (\beta^2-\gamma^2)(A^2-1)^2)]/[(\beta^2-\gamma^2)A^2(A^2-1)] \tag{11c}$$
$$\beta^2 - \gamma^2 = \alpha^2 A^2 \tag{11d}$$

and it possesses two special cases:

(1)

$$k_1 = 0 \qquad k_2 = 1$$

corresponding to

$$\phi_{3,1} = 4\tan^{-1}[A\sin(\alpha y)/\sinh(\beta z + \delta\gamma t)] \tag{12a}$$

with

$$\alpha = 1/[(A^2-1)^{1/2}] \tag{12b}$$
$$\gamma = [\beta^2 - A^2/(A^2-1)]^{1/2} \tag{12c}$$

(2)

$$k_1 = 1 \qquad k_2 = (1 - 1/A^4)^{1/4}$$

corresponding to

$$\phi_{3,2} = 4\tan^{-1}[A\tanh(\alpha y)\mathrm{cn}(\beta z + \delta\gamma t; k_1)/\mathrm{sn}(\beta z + \beta\gamma t; k_2)] \tag{13a}$$

with

$$\alpha = A/(A^2-1) \tag{13b}$$
$$\gamma = [\beta^2 - A^4/(A^2-1)^2]^{1/2}. \tag{13c}$$

Another possibility is presented by the wave

$$\phi_4 = 4\tan^{-1}[A\,\mathrm{sn}(\alpha y; k_1)\mathrm{sn}(\beta z + \delta\gamma t; k_2)/\mathrm{cn}(\alpha y; k_1)] \tag{14a}$$
$$k_1^2 = 1 - A^2[\alpha^2(1-A^2)-1]/[\alpha^2(1-A^2)] \tag{14b}$$
$$k_2^2 = A^2[1 + (\beta^2-\gamma^2)(1-A^2)]/[\beta^2-\gamma^2)(1-A^2)] \tag{14c}$$
$$\alpha^2 - (\beta^2-\gamma^2) = 1/(1-A^2). \tag{14d}$$

The special case here is

$$k_1 = (1 - 1/A^4)^{1/2} \qquad k_2 = 1$$

corresponding to

$$\phi_{4,1} = 4\tan^{-1}[A\,\mathrm{cn}(\alpha y; k_1)\tanh(\beta z + \delta\gamma t)/\mathrm{sn}(\alpha y; k_1)] \tag{15a}$$

with parameters

$$\alpha = 1/(1-A^2) \tag{15b}$$
$$\gamma = [\beta^2 - A^2/(A^2-1)^2]^{1/2}. \tag{15c}$$

The following possible wave is

$$\phi_5 = 4\tan^{-1}\{A\,\mathrm{sn}(\alpha y; k_1)/[\mathrm{cn}(\alpha y; k_1)\mathrm{sn}(\beta z + \delta\gamma t; k_2)]\} \tag{16a}$$
$$k_1^2 = 1 - A^2[1 + \alpha^2(A^2-1)]/[\alpha^2(A^2-1)] \tag{16b}$$

$$k_2^2 = [A^2 + (\beta^2 - \gamma^2)(A^2 - 1)]/[(\beta^2 - \gamma^2)A^2(A^2 - 1)] \tag{16c}$$

$$\beta^2 - \gamma^2 = \alpha^2 A^2. \tag{16d}$$

The possible special cases are:

(1)

$$k_1 = 1 \qquad k_2 = 0$$

corresponding to

$$\phi_{5,1} = 4\tan^{-1}[A\sinh(\alpha y)/\sin(\beta z + \delta\gamma t)] \tag{17a}$$

with

$$\alpha = [1/(1 - A^2)]^{1/2} \tag{17b}$$

$$\gamma = [\beta^2 - A^2/(1 - A^2)]^{1/2} \tag{17c}$$

(2)

$$k_1 = (1 - A^4)^{1/2} \qquad k_2 = 1$$

then

$$\phi_{5,2} = 4\tan^{-1}[A\mathrm{sn}(\alpha y; k_1)\tanh(\beta z + \delta\gamma t)/\mathrm{cn}(\alpha y; k_1)] \tag{18a}$$

with parameters

$$\alpha = 1/(A^2 - 1) \tag{18b}$$

$$\gamma = [\beta^2 - A^2/(A^2 - 1)^2]^{1/2}. \tag{18c}$$

One more complex wave is

$$\phi_6 = 4\tan^{-1}[A[\mathrm{sn}(\alpha y; k_1)\mathrm{sn}(\beta z + \delta\gamma t; k_2)]/[\mathrm{cn}(\alpha y; k_1)\mathrm{cn}(\beta z + \delta\gamma t; k_2)] \tag{19a}$$

$$k_1^2 = 1 - A^2[1 - \alpha^2(1 + A^2)]/[\alpha^2(1 + A^2)] \tag{19b}$$

$$k_2^2 = 1 - A^2[1 - (\beta^2 - \gamma^2)(A^2 + 1)]/[(\beta^2 - \gamma^2)(1 + A^2)] \tag{19c}$$

$$\alpha^2 + (\beta^2 - \gamma^2) = 1/(1 + A^2). \tag{19d}$$

The possible special cases here are:

(1)

$$k_1 = 0 \qquad k_2 = (1 - A^4)^{1/2}$$

corresponding to

$$\phi_{6,1} = 4\tan^{-1}[A\tan(\alpha y)\mathrm{sn}(\beta z + \delta\gamma t; k_2)/\mathrm{cn}(\beta z + \delta\gamma t; k_2)] \tag{20a}$$

with

$$\alpha = 1/(A^2 + 1) \tag{20b}$$

$$\gamma = [\beta^2 - 1/(A^2 + 1)^2]^{1/2} \tag{20c}$$

(2)

$$k_1 = (1 - A^4)^{1/2} \qquad k_2 = 0$$

corresponding to

$$\phi_{6,2} = 4\tan^{-1}[A\,\mathrm{sn}(\alpha y; k_1)\tan(\beta z + \delta\gamma t)/\mathrm{cn}[\alpha y; k_1] \tag{21a}$$

with

$$\alpha = 1/(A^2 + 1) \tag{21b}$$

$$\gamma = [\beta^2 - A^2/(A^2+1)^2]^{1/2}. \tag{21c}$$

The last wave is

$$\phi_7 = 4\tan^{-1}[A(\mathrm{cn}(\alpha y; k_1)\mathrm{sn}(\beta z + \delta\gamma t; k_2))/(\mathrm{sn}(\alpha y; k_1)\mathrm{cn}(\beta z + \delta\gamma t; k_2))] \tag{22a}$$

with parameters

$$k_1^2 = [\alpha^2(A^2+1)^2 - A^2]/[\alpha^2 A^2(A^2+1)] \tag{22b}$$

$$k_2^2 = 1 - A^2[1-(\beta^2-\gamma^2)(1+A^2)]/[(\beta^2-\gamma^2)(A^2+1)] \tag{22c}$$

$$\alpha^2 = (\beta^2-\gamma^2)A^2. \tag{22d}$$

Three special cases are possible here:

(1)

$$k_1 = 0 \qquad k_2 = (1-A^4)^{1/2}$$

then the wave is

$$\phi_{7,1} = 4\tan^{-1}[A\,\mathrm{cotan}\,(\alpha y)\mathrm{sn}(\beta z + \delta\gamma t; k_2)/\mathrm{cn}(\beta z + \delta\gamma t; k_2)] \tag{23a}$$

with parameters

$$\alpha = A/(A^2+1) \tag{23b}$$

$$\gamma = [\beta^2 - 1/(A^2+1)^2]^{1/2}. \tag{23c}$$

(2)

$$k_1 = k_2 = 1.$$

The wave in this case is

$$\phi_{7,2} = 4\tan^{-1}[A\sinh(\beta z + \delta\gamma t)/\sinh(\alpha y)] \tag{24a}$$

with parameters

$$\alpha = A/(A^2+1)^{1/2} \tag{24b}$$

$$\gamma = [\beta^2 - 1/(A^2+1)]^{1/2}. \tag{24c}$$

(3)

$$k_1 = (1-1/A^4)^{1/2} \qquad k_2 = 0.$$

The wave is

$$\phi_{7,3} = 4\tan^{-1}[A\,\mathrm{cn}(\alpha y; k_1)\tan(\beta z + \delta\gamma t)/\mathrm{sn}(\alpha y; k_1)] \tag{25a}$$

with parameters

$$\alpha = A^2/(A^2+1) \tag{25b}$$

$$\gamma = [\beta^2 - A^2/(A^2+1)^2]^{1/2}. \tag{25c}$$

3. Allowed areas for the wave parameters and the influence of the boundary conditions on the static structures

Because of the inequalities for the elliptic modules k_1 and k_2 there exist allowed and forbidden areas for the wavenumbers α, β wave frequency γ and wave amplitude A. The wave amplitude determines the nonlinearity of the wave and with respect to this quantity the following three possibilities exist for the discussed waves.

(1) $A < 1$. These waves can contain, as particular cases when A is infinitesimal, the correspondent solutions of the linear Klein–Gordon equation. An example for such a wave is ϕ_4.

(2) $A \geqslant 1$. Such waves are strongly nonlinear—they cannot be reduced to a linear wave. An example is presented in [27].

(3) Waves for which a forbidden area for the amplitude does not exist as in the case of the wave ϕ_7.

Below the parameters and static structures, connected to the wave ϕ_7, are mainly discussed. This wave is chosen because it contains as a particular case one of the vortex structures obtained by Borisov *et al* [28] and the single vortex structure, similar to this one, discussed by Hudak [29]. The allowed areas for the wavenumber and wave frequency are

$$\frac{A^2}{(A^2+1)^2} \leqslant \alpha^2 \leqslant \frac{A^2}{A^2+1} \tag{26}$$

$$\frac{A^2}{(A^2+1)^2} \leqslant \beta^2 - \gamma^2 \leqslant \frac{1}{1+A^2}. \tag{27}$$

Using the dispersion relation (22*d*) and inequalities (26) for α we obtain a second inequality for $\beta^2 - \gamma^2$

$$\frac{1}{(1+A^2)^2} \leqslant \beta^2 - \gamma^2 \leqslant \frac{1}{1+A^2} \tag{28}$$

which leads to a dependence of the bottom boundary of the allowed amplitude area on the wave amplitude. If $A < 1$ the bottom boundary is given by (28) and if $A \geqslant 1$ the bottom boundary is given by (27). Equations (27) and (28) show that there exists a minimum value of the wavenumber β under which the wave ϕ_7 does not exist. This minimum value depends on the wave amplitude and its behaviour with amplitude changes is different in the cases $A < 1$ and $A \geqslant 1$. The minimum value for β is reached when $\gamma = 0$, i.e. in the case of the structures discussed below.

One possible application of the obtained solutions of the $(2+0)$-dimensional sine–Gordon equation is the description of structures at temperatures below the critical temperature for the Kosterlitz–Thouless phase transitions in spin systems with an anisotropy magnetic or crystal field [28–33]. Let us have a spin system and $\psi(\bar{y}, \bar{z})$ is the angle of the spin at the $(\bar{y}, \bar{z})$ position with respect to some arbitrary axis. In this case the total energy of the slowly varying plane configurations in the correspondent spin system when also an anisotropy, created by an external magnetic field ($p = 1$) or by a crystal anisotropy field ($p = 2, 3, 4, 6$) exist, has the following representation

$$E = J \int\int \mathrm{d}\bar{y}\,\mathrm{d}\bar{z}[\nabla\psi(y, z)]^2 + J \int\int \mathrm{d}\bar{y}\,\mathrm{d}\bar{z}\frac{1-\cos(p\psi(\bar{y}, \bar{z}))}{p\lambda_p^2} \tag{29}$$

where $(J/p\lambda_p^2)$ is the anisotropy energy constant of the system. The spin configurations which extremalize the energy are those which solve the $(2+0)$-dimensional sine–Gordon equation

$$\frac{\partial^2\psi}{\partial\bar{y}^2} + \frac{\partial^2\psi}{\partial\bar{z}^2} = \frac{1}{p\lambda_p^2}\sin(p\psi) \tag{30}$$

and their vorticity can be calculated from

$$q = \frac{1}{2\pi}\oint_{\gamma^*}(\nabla\psi\cdot \mathrm{d}\boldsymbol{l}),\ \mathrm{d}\boldsymbol{l} = (\mathrm{d}\bar{y},\ \mathrm{d}\bar{z}) \tag{31}$$

where γ^* is an arbitrary loop in the $(\bar{y}, \bar{z})$-plane surrounding a point $(\bar{y}_0, \bar{z}_0)$ and anticlockwise oriented. If $\phi = p\psi$, $\bar{y} = p^{1/2}\lambda_p y$ and $\bar{z} = p^{1/2}\lambda_p z$ then (30) is reduced to the static case of (1).

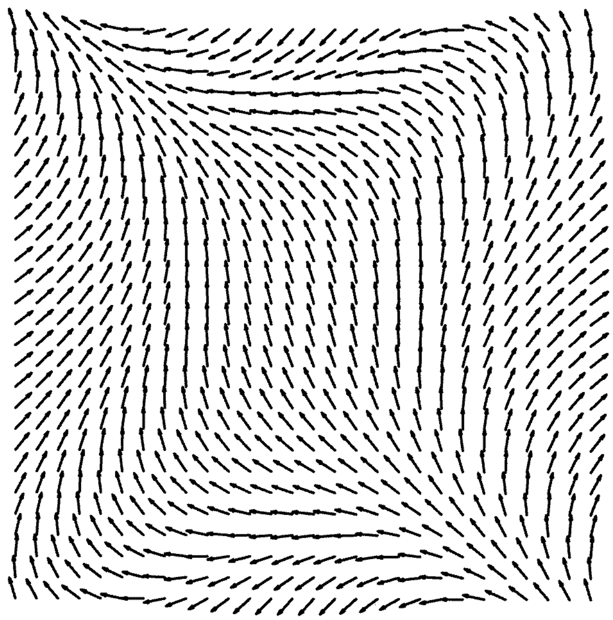

Figure 1. The structure connected to the solution $\phi^-_{1,1}$ $(\gamma = 0)$.

Let the system be treated as infinite, i.e. let the system sizes be much greater than the characteristic size connected to the problem. Then the aperiodic structures described above expressed by hyperbolic functions can exist. With respect to their vorticity they are structures having non-zero vorticity as for example $\phi_{7,2}(\gamma = 0)$, which has $q = 4$ in the case $p = 1$ or structures with zero vorticity as in the case of $\phi_{1,1}(\gamma = 0)$. The static structure, connected to the solution $\phi_{1,1}$ is presented in figure 1 and the structure connected to $\phi_{7,2}$ is presented in figure 2.

In general the solutions (2) describe two kinds of double-periodic structures, mentioned in the introduction. An example for the structures which have vanishing ϕ on the elementary cell boundaries is the structure connected to the wave $\phi_{7,3}$ (figure 3). When the system is infinite the parameters of the structures can have values which are within a continuum interval. When the system must be treated as finite, and if some boundary conditions are imposed, the structure parameters can have only discrete values. The combination between this property and the existence of forbidden areas for the parameters leads to one among

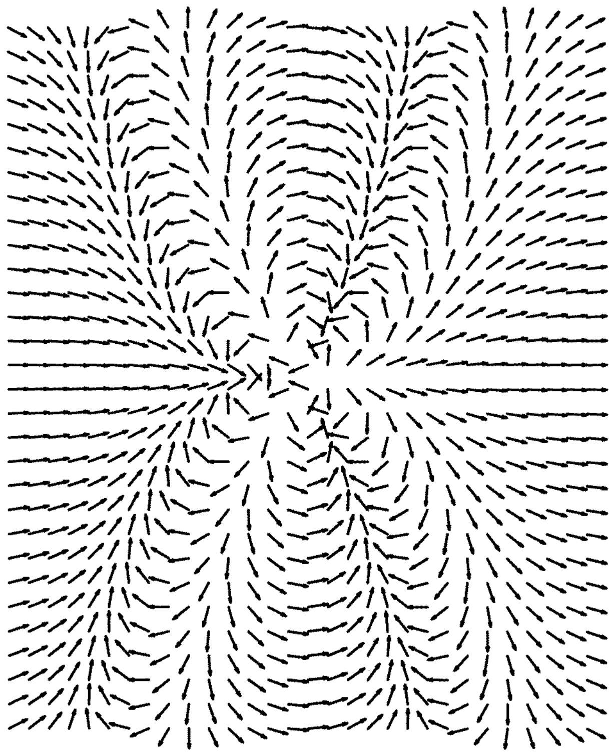

Figure 2. The static structure connected to the wave $\phi_{7,2}$.

the following possibilities (n and m are the number of the cells in the direction of the axes 0_y and 0_z).

(1) $n = 0$, $m = 0$. The system is small enough and all the values of the system parameters are in the forbidden areas—then the corresponding structure cannot exist in a system possessing such sizes. In other words some structures require for their existence a system with sizes greater than minimal ones.

(2) $n = 1$, $m = 1$. The system sizes allow the existence of only a single cell.

(3) $n = 1$, $m = 2, 3, 4, \ldots$. or $n = 2, 3, 4, \ldots$, $m = 1$. This case is characteristic for systems where one of the sizes is small and the another one is larger or much larger. In such systems band structures can exist consisting of one cell in the direction of one of the axes and of several cells in the direction of another axis.

(4) $n = 2, 3, 4, \ldots$, $m = 2, 3, 4, \ldots$. The sizes of the system are large enough and allow the existence of lattice structures.

Let us have a system in the area between $-a$ and a in the direction of the axis 0_y and between $-b$ and b in the direction of the axis 0_z. The imposition of boundary conditions on the structures has the consequence that the structure parameters become self-consistent: the spatial behaviour of the system in the direction of the axis 0_y influences the spatial behaviour in the direction of the axis 0_z and *vice versa*. Below the boundary condition of the kind

$$\phi = 0 \mid_{y=-a,a} \qquad \phi = 0 \mid_{z=-b,b} \tag{32}$$

is discussed.

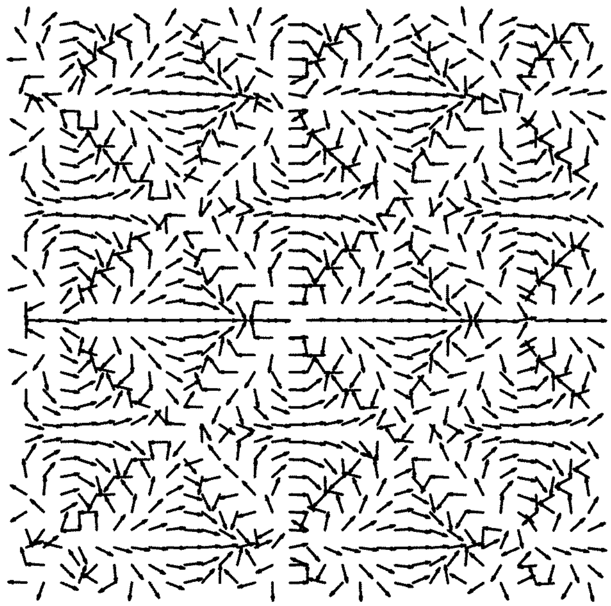

Figure 3. The static structure connected to the wave $\phi_{7,3}$.

With respect to the structures connected with ϕ_7 condition (32) leads to the equalities

$$\left.\frac{\partial\phi}{\partial y}\right|_{z=-b,b} = 0 \qquad \left.\frac{\partial\phi}{\partial z}\right|_{y=-a,a} = 0. \tag{33}$$

The latter boundary conditions and equalities are satisfied on the boundaries of the elementary cell of the structure and on the system boundaries. Let the loop γ^* coincide with the boundaries of the elementary cell or with the boundaries of the system. Then the total vorticity of the structures included in the elementary cell and the total vorticity of the structures discussed below is $q = 0$.

The boundary conditions (32) lead to the following equations for the discrete self-consistent system parameters

$$\alpha_{nm} = \frac{2n+1}{a} K(k_{1nm}) \tag{34}$$

$$\beta_{nm} = \frac{2m}{b} K(k_{2nm}) \tag{35}$$

$$A_{nm}^2 = \frac{(2n+1)^2 b^2}{4m^2 a^2} \frac{K^2(k_{1nm})}{K^2(k_{2nm})} \tag{36}$$

$$k_{1nm}^2 = \frac{\alpha_{nm}^2 (A_{nm}^2 + 1)^2 - A_{nm}^2}{\alpha_{nm}^2 A_{nm}^2 (A_{nm}^2 + 1)} \tag{37}$$

$$k_{2nm}^2 = 1 - \frac{A_{nm}^2 [1 - \beta_{nm}^2 (1 + A_{nm}^2)]}{\beta_{nm}^2 (A_{nm}^2 + 1)} \tag{38}$$

where $K(k_i)$, $i = 1, 2$, is the complete elliptic integral of first kind and n and m can have values $1, 2, 3, \ldots$.

The inequalities for the elliptic modules give the following two cases of inequalities for the parameters α and β when the amplitude $A \geqslant 1$.

Case A

$$\alpha_{nm} \geqslant \tfrac{1}{2} \tag{39}$$

$$\beta_{nm} \geqslant \tfrac{1}{2} \tag{40}$$

$$\alpha_{nm}^2 + \beta_{nm}^2 \leqslant 1. \tag{41}$$

From the inequality (41) inequalities for the numbers n and m follow if the system sizes a and b are fixed and if the elliptic integral of the first kind obtains their minimum value $\pi/2$:

$$1 \leqslant n \leqslant \frac{1}{2}\left(\frac{2a}{\pi}\sqrt{1 - m^2\frac{\pi^2}{b^2}} - 1\right) \tag{42}$$

$$1 \leqslant m \leqslant \frac{b}{\pi}\sqrt{1 - (2n+1)\frac{\pi^2}{4a^2}}. \tag{43}$$

Case B

$$0 < \alpha_{nm} \leqslant \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \beta_{nm}^2} \tag{44}$$

$$0 < \beta_{nm} \leqslant \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \alpha_{nm}^2} \tag{45}$$

from which the inequalities for the numbers n and m are

$$1 \leqslant n < \frac{1}{2}\left(\frac{a}{\pi} - 1\right) \tag{46}$$

$$1 \leqslant m < \frac{b}{2\pi}. \tag{47}$$

If the amplitude A is smaller than 1 in case A we have inequalities (39) and (41) and in case B the inequalities are $\alpha < \frac{1}{2}$ and (45). Simple calculations for the minimum system sizes required by the structures satisfying the inequalities of case B show that

(1) if the system sizes are $a < 3\pi$, $b < 2\pi$ the structures cannot exist;

(2) a single structure can exist if $a > 3\pi$, $b > 2\pi$;

(3) simple band structures can exist if $a > 3\pi$, $b > 4\pi$ $(n = 1, m = 2)$, or when $a > 5\pi$, $b > 2\pi$ $(n = 2, m = 1)$;

(4) the minimum sizes for the existence of a lattice structure $(n = 2, m = 2)$ are $a > 5\pi$, $b > 4\pi$.

Using the system of equations (34)–(38) the parameters of the self-consistent structures satisfying the inequalities of case A can be calculated numerically. The results of these calculations are presented in table 2.

4. Conclusions

The investigations show that the sine–Gordon waves and structures obtained by the approach developed in [20] possess some general features. With respect to the static structures there exist two possible kinds of structures exhibiting different behaviour of ϕ on the boundaries of the elementary cell of the structure. Two similar kinds of structures can also exist in positive- and negative-temperature Poisson–Boltzmann systems [34, 35]. When appropriate boundary conditions are applied the static structures require minimal system sizes for their existence. This feature is connected to the modules of the Jacobi elliptic functions which

Table 2. The self-consistent parameters for some cell structures connected to the solution ϕ_7.

n	m	a	b	α	β	A	k_1^2	k_2^2
1	1	9.4639	6.2978	0.5111	0.5443	0.9391	0.1	0.3
1	1	9.6904	6.3892	0.5306	0.6103	0.8694	0.3	0.6
1	2	9.4639	12.5957	0.5111	0.5443	0.9391	0.1	0.3
1	3	9.4411	18.8707	0.5124	0.5277	0.9710	0.1	0.2
2	2	16.9988	13.0705	0.4743	0.6908	0.6866	0.1	0.8
2	3	15.7446	18.8935	0.5270	0.5270	1.0	0.2	0.2
3	2	25.7385	13.5769	0.4385	0.7596	0.5774	0.1	0.9
3	3	22.5715	19.0970	0.4934	0.6125	0.8055	0.05	0.6
3	4	22.3222	25.3208	0.4932	0.5858	0.8420	0.005	0.5
4	3	30.5950	19.6013	0.4680	0.6909	0.6774	0.05	0.8
4	4	28.2805	25.1356	0.5011	0.5104	0.9818	0.01	0.08
5	4	43.4638	28.0939	0.4001	0.8032	0.4981	0.025	0.94
5	5	34.5691	31.4214	0.5017	0.5129	0.9782	0.015	0.098

must have discrete values because of the boundary conditions. But the modules determine the periods of the structure and thus the system must have enough large sizes in order to satisfy the boundary conditions with the corresponding values of the structure periods. With respect to the waves, there exist allowed and forbidden areas for the wavenumbers and wave frequencies as for the waves with $v < v_l$ and for the waves with $v > v_l$. The mathematical reasons for the origin of such areas are the inequalities for the modules of the Jacobi elliptic functions. The imposition of the stability requirement makes the situation more complex. In general the area of stability is not the same as the allowed area for the wave parameters. The investigation of the relation between the stability area and the allowed areas for the wave parameters of the discussed waves as well as their applications for the description of processes in large two-dimensional Josephson junctions, ferromagnetic systems and dynamics of the crystal lattices is a subject of current research and will be presented in future papers.

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